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Integration of asymptotically almost periodic functions and weak asymptotic almost periodicity

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CONTENTS

Summary

We resolve the question as to when an asymptotically almost periodic function with values in a Banach space has an asymptotically almost periodic integral. Further, we characterize the weakly relatively compact sets in spaces of vector valued continuous functions and use the resulting criteria to investigate the relationship between weak almost periodicity in the sense of Eberlein and the parallel notion utilized by Amerio in his approach to extending the classical Bohl–Bohr integration theorem. One consequence is the discovery of an unexpected connection between Eberlein weak almost periodicity and the asymptotic almost periodicity of integrals. We also apply our methods to investigate periodicity properties inherited by integrals of functions which are only asymptotically almost periodic in a weak sense.

0. Introduction

The Bohr–Neugebauer theorem (cf. [21]), for example, addresses a problem of longstanding interest in the theory of differential equations by providing an instance in which asymptotic behavior of solutions can be predicted from corresponding properties inherent in the equation; namely, in the case of an ordinary linear differential equation

$$\text{(0.a)} \qquad \sum_{k=0}^{n} a_k y^{(k)}(t) = f(t), \qquad t \in \boldsymbol{R},$$

with constant coefficients, if f is a scalar valued almost periodic function, then every bounded solution of (0.a) is also almost periodic. Indeed, this general problem has been treated with notable success in the context of almost periodic equations, and we refer to the monographs [1] and [21] for a wealth of information on the subject. The key to these developments, however, is a result which resolves an extremely basic form of the general question – the classical Bohl–Bohr integration theorem (cf. [21]) asserts that the indefinite integral $F(t) = \int_0^t f(u)\,du$ of a complex valued almost periodic function f defined on $\boldsymbol{R}$ will as well be almost periodic on $\boldsymbol{R}$ whenever F is bounded.

Although many results concerning scalar almost periodic functions can be directly carried over to the case of almost periodic functions taking values in an arbitrary Banach space X, the Bohl–Bohr theorem is not among them (cf. [1, p. 53]). Nonetheless, L. Amerio (cf. [1]) has shown that a verbatim version of this important result does hold in case the Banach space X is uniformly convex, while M.I. Kadets [18] has subsequently given the best possible extension by showing that the Bohl–Bohr–Amerio theorem remains valid as long as X does not contain an isomorphic copy of c_0. These results, in turn, have led to a development for almost periodic equations in Banach spaces which parallels that in the scalar case, and we again refer to [1] and [21] for details.

The basic question resolved by Kadets' generalized Bohl–Bohr theorem has a natural counterpart in the context imposed by equations describing only a forward evolution in time. In particular, when attention is restricted to

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a given halfline $J_a = [a, \infty)$, $a \in \boldsymbol{R}$, the part of the almost periodic functions on $\boldsymbol{R}$ with values in a Banach space X is taken by the space $AAP(J_a, X)$ of all X-valued *asymptotically* almost periodic functions defined on J_a (cf. [12], [13] and [25], [26]), and the question becomes the following:

(0.b) *For* $f \in AAP(J_a, X)$ *and* $F(t) = \int_a^t f(u)\,du$, $t \in J_a$, *under what conditions is it true that* $F \in AAP(J_a, X)$?

A partial response to the problem in this setting has been given by M. Fréchet [12], [13], who established a form of Bohl–Bohr theorem for asymptotically almost periodic functions with finite dimensional range, but (0.b) has remained otherwise unsettled. Our work in the sequel provides a complete solution.

In the course of our investigation, we were led to consider a weak notion of asymptotic almost periodicity which is directly related to a concept originally introduced by W.F. Eberlein [10] in the context of scalar valued continuous functions on abelian locally compact topological groups. To be more precise, taking $C_b(J_a, X)$ to denote the space of bounded continuous functions from a halfline J_a into a Banach space X and assuming that $C_b(J_a, X)$ is equipped with the topology of uniform convergence, the idea that entered the picture involves those $f \in C_b(J_a, X)$ such that the corresponding set $H^+(f) = \{f_\omega \colon \omega \in \boldsymbol{R}^+\}$ of translates is weakly relatively compact in $C_b(J_a, X)$, and this view of weak almost periodicity brought us to the problem of developing suitable characterizations of weak relative compactness in spaces of vector valued continuous functions. The criteria that evolved have been instrumental in our subsequent discovery of an unexpected connection between weak almost periodicity in the sense of Eberlein and the asymptotic almost periodicity of integrals, as well as in treating other questions which have arisen at various stages along the way.

Although from a different perspective, the idea of characterizing weak compactness in the setting of vector valued functions had first suggested itself in connection with our work in [25] to extend the classical Arzelà–Ascoli theorem. There, for reasons which equally apply to the question of weak compactness, we choose to develop our characterizations of precompact sets within the framework provided by spaces of vector valued continuous functions with topologies induced by weighted analogues of the supremum norm, and we now take a parallel path in our approach to the problem at hand. Following a brief preliminary section in which we set the context for our study of weak compactness, we begin in Section 2 by establishing our general criteria. In Section 3, the focus is on weak asymptotic almost periodicity, and we take advantage of our work in the preceding section to develop criteria and collect examples which we will require for our subsequent discussion of integrals. Moreover, addressing a question raised in [25], we shed some light on the distinction between weak almost periodicity in the sense of Eberlein and a second widely studied concept of weak almost periodicity (cf. [1]) which Amerio had utilized in his approach to the almost periodic version of (0.b). The fourth and final section then centers on our solution of (0.b), but we also apply our methods to investigate related questions about integrals of functions which are only asymptotically almost periodic in a weak sense.

Some of the results of this paper have been announced in [27].

1. Preliminaries

A detailed discussion of the setting in which we shall develop our characterizations of the weakly relatively compact sets can be found in our earlier treatise [25] on compactness and asymptotic almost periodicity. Here, consequently, we only pause to set the notation and recall some requisite ideas.

Throughout the remainder of the article, we let T denote a completely regular Hausdorff space. A nonnegative upper semicontinuous function on T will be called a *weight* (on T). If V is a set of weights on T such that, given any $t \in T$, there is some $v \in V$ for which $v(t) > 0$, we write $V > 0$. A set V of weights on T is said to be *directed upward* provided that, for every pair $v_1, v_2 \in V$ and each $\lambda > 0$, there exists $v \in V$ so that $\lambda v_i \leqslant v$ (pointwise on T) for $i = 1, 2$. Since there is no loss of generality, we hereafter assume that sets of weights are directed upward; a set V of weights on T which additionally satisfies $V > 0$ will be referred to as a *system of weights* on T.

The phrase "locally convex space" will henceforth be taken to mean a Hausdorff locally convex topological vector space over $\boldsymbol{K} \in \{\boldsymbol{R}, \boldsymbol{C}\}$; there will be no loss of generality in tacitly assuming that $\boldsymbol{K} = \boldsymbol{C}$. Further, the set of all continuous seminorms on a locally convex space X will be denoted by $cs(X)$, while we write $C(T, X)$ to indicate the collection of all continuous functions from T into X. Now, taking a system V of weights on T and a locally convex space X, we consider the following vector spaces (over $\boldsymbol{K}$) of continuous functions associated with the triple (T, V, X):

$$CV_0(T, X) = \{f \in C(T, X)\colon\ vf \text{ vanishes at infinity on } T \text{ for all } v \in V\};$$

$$CV_{\mathrm{p}}(T, X) = \{f \in C(T, X)\colon\ vf(T) \text{ is precompact in } X \text{ for all } v \in V\};$$

$$CV_{\mathrm{b}}(T, X) = \{f \in C(T, X)\colon\ vf(T) \text{ is bounded in } X \text{ for all } v \in V\}.$$

Obviously, $CV_{\mathrm{p}}(T, X) \subseteq CV_{\mathrm{b}}(T, X)$, while the upper semicontinuity of the weights yields that $CV_0(T, X) \subseteq CV_{\mathrm{p}}(T, X)$. Thus, if for each $v \in V$, $q \in cs(X)$, and $f \in C(T, X)$, we put

$$p_{v,q}(f) = \sup\{v(t)q(f(t))\colon\ t \in T\},$$

then $p_{v,q}$ can be regarded as a seminorm on either $CV_{\mathrm{b}}(T, X)$, $CV_{\mathrm{p}}(T, X)$, or $CV_0(T, X)$; we assume that each of these three spaces is equipped with the Hausdorff locally convex topology induced by $\{p_{v,q}\colon\ v \in V, q \in cs(X)\}$.

In case $X = \mathbf{K}$, we will omit X from our notation and write, say, $CV_0(T)$ in place of $CV_0(T, \mathbf{K})$; we also then put $p_v = p_{v,q}$ for each $v \in V$, where $q(z) = |z|$, $z \in \mathbf{K}$. Similarly, if $X = (X, q)$ is any normed space and $v \in V$, we write p_v instead of $p_{v,q}$. As a matter of further notational convenience, given $v \in V$ and $q \in cs(X)$, the closed unit ball corresponding to the seminorm $p_{v,q}$ in either $CV_0(T, X)$, $CV_p(T, X)$, or $CV_b(T, X)$ will be denoted by $B_{v,q}$, or simply B_v in case $X = (X, q)$ is a normed space; this ambiguity should occasion no difficulty since the setting under consideration will always be clear from context.

Aside from examples that arise in response to special considerations such as those inherent in the dynamics of age dependent populations (cf. [34]), many standard spaces of continuous functions can be realized in the format set forth above, and we shall here list certain instances for future references. To this end, given a completely regular Hausdorff space T, and writing 1_F to designate the characteristic function of a subset F of T, we distinguish three systems of weights on T: namely,

$$\mathscr{K} = \mathscr{K}(T) = \{\lambda 1_K \colon \lambda > 0,\ K \subseteq T,\ K \text{ compact}\}, \qquad 1 = 1(T) = \{\lambda 1_T \colon \lambda > 0\},$$

and the system $U = U(T)$ consisting of all weights on T which vanish at infinity. Further, given a locally convex space X, we put

$$C_0(T, X) = \{f \in C(T, X) \colon f \text{ vanishes at infinity on } T\},$$

$$C_p(T, X) = \{f \in C(T, X) \colon f(T) \text{ is precompact in } X\},$$

$$C_b(T, X) = \{f \in C(T, X) \colon f(T) \text{ is bounded in } X\}.$$

EXAMPLE 1.1. For any pair (T, X) consisting of a completely regular Hausdorff space T and a locally convex space X,

(1.a) $$C\mathscr{K}_0(T, X) = C\mathscr{K}_p(T, X) = C\mathscr{K}_b(T, X) = (C(T, X), \varkappa),$$

where $\varkappa$ denotes the compact-open topology;

(1.b) $$CU_0(T, X) = CU_p(T, X) = CU_b(T, X) = (C_b(T, X), \beta_0),$$

where β_0 denotes the substrict topology (cf. [11]);

(1.c) $$C1_0(T, X) = (C_0(T, X), \nu) C1_p(T, X) = (C_p(T, X), \nu),$$
$$C1_b(T, X) = (C_b(T, X), \nu),$$

where we use ν in each case to denote the topology of uniform convergence on T.

The use of functional analytic techniques to link the vector and scalar cases is an important factor in our study of compactness, and the idea on which we base this aspect of our approach follows from a linearization principle for vector valued functions that is incorporated into the notion of ε-product (in the sense of L. Schwartz [29], [30]). In our context, the question of ε-product representation has been considered by K.-D. Bierstedt [3]. Before stating Bierstedt's result in this direction, however, we first present one additional item of terminology.

A completely regular Hausdorff space T is said to be a $V_{\mathbf{R}}$*-space* with respect to a given system V of weights on T if a function $f: T \to \mathbf{R}$ is necessarily continuous whenever, for each $v \in V$, the restriction of f to $\{t \in T: v(t) \geqslant 1\}$ is continuous. This requirement on a pair (T, V) can essentially be construed as a condition for the completeness of $CV_0(T)$ (cf. [15]), and we note in passing that it is a relatively modest restriction. Indeed, if $V = \mathcal{K}(T)$, the only requirement is that T be a $k_{\mathbf{R}}$-space (in the language of H. Buchwalter), which certainly holds, say, when T is locally compact, while no restriction whatsoever is imposed on T in case $V \doteq 1(T)$.

THEOREM 1.2. (Bierstedt [3, p. 39]). *Let V be a system of weights on a completely regular Hausdorff space T, and assume that X is a quasicomplete locally convex space. If T is a $V_{\mathbf{R}}$-space, then*

$$CV_0(T, X) \cong X\varepsilon CV_0(T) \quad \text{and} \quad CV_p(T, X) \cong X\varepsilon CV_b(T).$$

2. Weak compactness

In developing our criteria for weak relative compactness, we impose the same standing hypothesis under which we established our results of Arzelà–Ascoli type in [25]—viz.,

(2.a) T is a completely regular Hausdorff space, V is a system of weights on T, and T is a $V_{\mathbf{R}}$-space such that, for each $t \in T$, there exists $f_t \in CV_0(T)$ with $f_t(t) \neq 0$;

these basic assumptions will be in force throughout the present section. Also, given $v \in V$, let us put $N(v) = \{t \in T: v(t) > 0\}$.

The point of departure for our work is the classical result due to A.Grothendieck [16] in which he characterizes the weakly relatively compact subsets of $C1_b(T) = (C_b(T), \nu)$ in terms of the interchangeable double limits condition. From the lead set in [25], we expected from the outset that this criterion should carry over to spaces of type $CV_p(T, X)$. However, since weakly asymptotically almost periodic functions with precompact range are already asymptotically almost periodic [25, Theorem 3.6], something more was required for the applications we had in mind. We now proceed to show that Grothendieck's interchangeable double limits characterization can be extended to the class of spaces of type $CV_b(T, X)$.

THEOREM 2.1. *Let X be a quasicomplete locally convex space. A subset H of $CV_b(T, X)$ is then weakly relatively compact if and only if H is bounded in $CV_b(T, X)$ and the following interchangeable double limits property holds for every $v \in V$ and each q in a given generating family $\mathcal{S}$ of continuous seminorms on X: for all sequences (h_m) in H, (t_n) in $N(v)$, and (x'_n) in* $\mathrm{ext}(B_q^\circ)$,

$$\lim_m \lim_n \langle v(t_n)h_m(t_n), x'_n\rangle = \lim_n \lim_m \langle v(t_n)h_m(t_n), x'_n\rangle$$

whenever both iterated limits exist.

Remark. As the proof will show, the part of $\mathrm{ext}(B_q^\circ)$ can be taken by any subset A of B_q° having the property that its $\sigma(X', X)$-closed absolutely convex hull coincides with B_q°. (Of course, following our convention for denoting the unit ball associated with a seminorm, $B_q = \{x \in X:\ q(x) \leqslant 1\}$, while $\mathrm{ext}(B_q^\circ)$ denotes the set of extreme points of the polar of B_q in X'.)

Proof. We shall reduce the general case to the classical situation treated by Grothendieck. To this end, for $v \in V$ and $q \in \mathscr{S}$, let us equip $\mathrm{ext}(B_q^\circ)$ and $E_v = \{v(t)\delta_t:\ t \in N(v)\}$ with the topologies induced by the weak-star topology $\sigma(X', X)$ and the strong topology $\beta(CV_b(T)', CV_b(T))$, respectively. Next, given $f \in CV_b(T, X)$, consider the corresponding function $f_{v,q}$: $E_v \times \exp(B_q^\circ) \to K$ defined by $f_{v,q}(v(t)\delta_t, x') = \langle v(t)f(t), x'\rangle$. The fact that $\{x' \circ f:\ x' \in B_q^\circ\}$ is a bounded subset of $CV_b(T)$ readily yields that $f_{v,q} \in C_b(E_v \times \mathrm{ext}(B_q^\circ))$. Moreover,

$$\|f_{v,q}\| = \sup\{|f_{v,q}(v(t)\delta_t, x')|:\ t \in N(v),\ x' \in \mathrm{ext}(B_q^\circ)\}$$
$$\leqslant \sup\{|\langle v(t)f(t), x'\rangle|:\ t \in T,\ x' \in B_q^\circ\} = p_{v,q}(f).$$

If $\|f_{v,q}\| < p_{v,q}(f)$, there would exist $(t, y') \in N(v) \times B_q^\circ$ such that

$$\text{(2.b)} \qquad |\langle v(t)f(t), x'\rangle| \leqslant \|f_{v,}\| < |\langle v(t)f(t), y'\rangle|$$

for all $x' \in \mathrm{ext}(B_q^\circ)$. However, since the inequality (2.b) would then hold as well for all x' belonging to the $\sigma(X', X)$-closed absolutely convex hull of $\mathrm{ext}(B_q^\circ)$, this would stand in contradiction to the Krein–Milman theorem, and so we conclude that $\|f_{v,q}\| = p_{v,q}(f)$. Setting $\Phi(f)(v, q) = f_{v,q}$ for $f \in CV_b(T, X)$, $v \in V$, and $q \in \mathscr{S}$, the map

$$\Phi:\ CV_b(T, X) \to \Pi\{C1_b(E_v \times \mathrm{ext}(B_q^\circ)):\ v \in V,\ \ q \in \mathscr{S}\}$$

is thus a topological linear embedding. Consequently, because the quasicompleteness of X implies that $CV_b(T, X)$ is quasicomplete, a bounded subset H of $CV_b(T, X)$ will be weakly relatively compact if, and only if, $\Phi(H)(v, q)$ is weakly relatively compact in $C1_b(E_v \times \mathrm{ext}(B_q^\circ))$ for each $(v, q) \in V \times \mathscr{S}$, and the desired equivalence is now immediate from [16, Théorème 6].

Although the interchangeable double limits condition specified in Theorem 2.1 may not appear too tractable at first sight, we will demonstrate in the subsequent sections that this criterion can indeed be used to advantage as a tool for investigating vector valued weakly almost periodic functions in the sense of Eberlein [10]. Furthermore, keeping in mind the classical case of $C1_b(T)$, there is not much hope for finding a more amenable characterization even in the setting provided by spaces of type $CV_p(T, X)$. For spaces of type $CV_0(T, X)$, however, we can give an alternative description of the weakly relatively compact subsets in terms of pointwise-weak relative compactness, and we now proceed to do this in the following extension of another classical result due to Grothendieck [16, Théorème 5]. As a matter of notational convenience in what follows, when given a generating family $\mathscr{S}$ of continuous seminorms on a locally convex space X, we will let $e_{\mathscr{S}}(X)$ denote the topology of pointwise convergence on $\bigcup\{\mathrm{ext}(B_q^\circ)\colon q\in\mathscr{S}\}$.

THEOREM 2.2. *Let X be a quasicomplete locally convex space. Then the following are equivalent for a subset H of $CV_0(T, X)$:*

1. *H is weakly relatively compact in $CV_0(T, X)$;*
2. (i) *H is bounded in $CV_0(T, X)$, and*
   (ii) *given a generating family $\mathscr{S}$ of continuous seminorms on X, H is pointwise $-$ $e_{\mathscr{S}}(X)$ relatively countably compact in $CV_0(T, X)$;*
3. (i) *H is bounded in $CV_0(T, X)$,*
   (ii) *for every $t\in T$, $H(t)$ is weakly relatively compact in X (equivalently, $e_{\mathscr{S}}(X)$ relatively countably compact for any generating family $\mathscr{S}$ of continuous seminorms on X), and*
   (iii) *if $f\colon T\to X$ is the pointwise-weak limit of a net in H, then $f\in CV_0(T, X)$.*

The proof of Theorem 2.2 will be divided into several steps, and we begin by noting the recent result by J. Bourgain and M. Talagrand [5, Théorème 1] in which the weak compactness of any bounded subset of a Banach space is equated with countable compactness relative to the topology of simple convergence on the extreme points of the dual unit ball. In the obvious way, this result can be extended to show that a bounded subset of any quasicomplete locally convex space X is weakly relatively compact if, and only if, it is $e_{\mathscr{S}}(X)$ relatively countably compact for an arbitrary generating family $\mathscr{S}$ of continuous seminorms on X, and so the task at hand is to suitably characterize the extreme points of the sets $(B_{v,q})^\circ$ in $CV_0(T, X)'$ for all $v\in V$ and $q\in cs(X)$.

THEOREM 2.3. *Let X be a quasicomplete locally convex space. If $v\in V$, $q\in cs(X)$, and $B_{v,q}$ is the corresponding neighborhood of zero in $CV_0(T, X)$, then*

$$\mathrm{ext}(B_{v,q}^\circ)=\{v(t)\delta_t\otimes x'\colon\ t\in N(v),\ x'\in\mathrm{ext}(B_q^\circ)\}.$$

To prove 2.3, since $CV_0(T, X)\cong X\varepsilon CV_0(T)$ by Theorem 1.2, [24, Theorem 2.2] applies to give us that

$$\mathrm{ext}(B_{v,q}^\circ)=\mathrm{ext}(B_v^\circ)\otimes\mathrm{ext}(B_q^\circ).$$

The problem is thus reduced to a corresponding one for the scalar case, and the following lemma will serve to complete the argument.

LEMMA 2.4. *For $v\in V$, if B_v is the corresponding zero neighborhood in $CV_0(T)$, then*

(i) $\mathrm{ext}(B_v^\circ)=\{\alpha v(t)\delta_t\colon\ t\in N(v),\ \alpha\in\boldsymbol{K}$ *with* $|\alpha|=1\}$, *and*

(ii) $E_v=\{\alpha v(t)\delta_t\colon\ t\in N(v),\ \alpha\in\boldsymbol{K}$ *with* $|\alpha|\leqslant 1\}$ *is weak-star compact in* $CV_0(T)'$.

Remark. The representation 2.4 (i) includes the well-known characterization of the extreme points in the unit ball of $C1_0(T)'$ which R.F. Arens and

J.L. Kelley [2] established in the case that T is compact. For comparison of 2.4(i) with more recent special cases, we refer to [32, p. 329] and [15, p. 157].

Proof of Lemma 2.4. Given $t \in N(v)$, if $\varepsilon \in (0, 1)$, then it is an easy consequence of [22, Lemma 2, p. 69] that there is a function $f_\varepsilon \in CV_0(T)$ such that $f_\varepsilon(T) \subseteq \mathbf{R}^+$, $f_\varepsilon \in B_v$, and $1-\varepsilon < v(t)f(t) < 1$. Using this fact, and noting that every $g \in CV_0(T)$ then has the form $g = h+g(t)[f_\varepsilon(t)^{-1}f_\varepsilon]$ where $h \in CV_0(T)$ with $h(t) = 0$, one can readily adapt the argument in the appropriate direction from the proof of the classical Arens–Kelley result as given by G. Köthe [19, p. 334] to show that the functional $\alpha v(t)\delta_t \in \operatorname{ext}(B_v^\circ)$ whenever $t \in N(v)$ and $\alpha \in \mathbf{K}$ with $|\alpha| = 1$. For the reverse inclusion, as well as to verify (ii), let T_d denote T with the discrete topology, equip $C(\beta T_d)$ with the topology induced by the supremum norm, and consider the continuous linear map l_v: $CV_0(T) \to C(\beta T_d)$ defined by $l_v(f) = \widetilde{vf}$ where $\widetilde{vf}$ is the (unique) continuous extension of vf from T_d to the Stone–Čech compactification βT_d. If we let B_1 denote the closed unit ball in the normed space $C(\beta T_d)$, then $l_v^{-1}(B_1) = B_v$ whence $B_v^\circ = l_v'(B_1^\circ)$, and therefore $\operatorname{ext}(B_v^\circ) \subseteq \{l_v'(\alpha\delta_p)\colon p \in \beta T_d,\ |\alpha| = 1\}$ (cf. [19, pp. 333–334]); we claim that $\{l_v'(\alpha\delta_p)\colon p \in \beta T_d,\ |\alpha| \leqslant 1\} \subseteq E_v$. To see this, consider $p \in \beta T_d \backslash T$ such that $l_v'(\delta_p) \neq 0$. Choosing $f \in CV_0(T)$ with $l_v(f)(p) \neq 0$, put

$$K = \{s \in T\colon\ v(s)|f(s)| \geqslant \tfrac{1}{2}|l_v(f)(p)|\}.$$

Next, taking a net (t_λ) in T which converges to p in βT_d, we note that (t_λ) is eventually in the compact set K since $v(t_\lambda)f(t_\lambda) = l_v(f)(t_\lambda)$ converges to $l_v(f)(p)$, and therefore (t_λ) clusters to some $t \in K$. It thus follows that $t \in N(v)$ and, given any $g \in CV_0(T)$, because $v|g|$ is upper semicontinuous,

$$|\langle g,\ l_v'(\delta_p)\rangle| = |l_v(g)(p)| \leqslant v(t)|g(t)| = |\langle g,\ v(t)\delta_t\rangle|.$$

But this implies (cf. [17, p. 186]) that there is some $\alpha \in \mathbf{K}$ such that $l_v'(\delta_p) = \alpha v(t)\delta_t$, as well as that $|\alpha| \leqslant 1$, which serves to establish our claim. (Indeed, $\alpha \in (0, 1]$, and we would even have that $\alpha = 1$ in case $v|K \in C(K)$.) Having now shown that

$$\{\alpha v(t)\delta_t\colon\ t \in N(v),\ |\alpha| = 1\} \subseteq \operatorname{ext}(B_v^\circ) \subseteq E_v,$$

we immediately conclude that (i) holds since a functional $\alpha v(t)\delta_t$ from E_v clearly fails to be an extreme point of B_v° whenever $|\alpha| < 1$. Moreover, we have that E_v is the image under l_v' of $\{\alpha\delta_p\colon\ p \in \beta T_d,\ |\alpha| \leqslant 1\}$, which means that (ii) must also hold since this latter set is homeomorphic to $\{\alpha \in \mathbf{K}\colon\ |\alpha| \leqslant 1\} \times \beta T_d$, and so the proof is complete. ∎

Remark. In the setting of Lemma 2.4, if $v|N(v)$ happens to be continuous, then the above argument shows that both $\operatorname{ext}(B_v^\circ) \cup \{0\}$ and $\{v(t)\delta_t\colon\ t \in N(v)\} \cup \{0\}$ would also be weak-star compact subsets of $CV_0(T)'$.

At this point, in order to prove Theorem 2.2, we need only fit together the collected pieces.

Proof of Theorem 2.2. The equivalence of 1 and 2 follows from Theorem 2.3 and the aforementioned extension of the Bourgain–Talagrand result. Since 1 clearly implies 3, it will be enough to show that 3 implies 2. To this end, assuming that 3 holds, let us fix a net (h_λ) in H. As follows from 3(ii), there is a subnet of (h_λ) which is pointwise-weak convergent to a function $f\colon T\to X$. But then, according to 3(iii), $f\in CV_0(T, X)$ whereby H is a pointwise-weak relatively compact subset of $CV_0(T, X)$ so that 2(ii) holds, and this serves to conclude the argument. ∎

Some aspects of Theorem 2.2 have been considered in [8]. There, for example, it was shown that a bounded *sequence* in $CV_0(T, X)$ is weakly convergent to a function $f\in CV_0(T, X)$ if, and only if, f is the pointwise-weak limit of the sequence [8, Theorem 4.2]. Moreover, the equivalence of conditions 1 and 3 from 2.2 was established in certain special cases [8, Theorem 4.3].

In bringing this section to a close, we pause to note one immediate application of Theorem 2.2. When T is compact, $C(T) = C1_0(T)$ is reflexive only if T is finite, and direct analogues of this standard fact have also been established in the settings of the compact-open and strict topologies by S. Warner [33] and H.S. Collins [7], respectively. By combining Theorem 2.2 with our characterization of the precompact sets in spaces of type $CV_0(T, X)$ [25, Theorem 2.1], we can extend these results to any space of type $CV_0(T)$ in the general class presently under consideration.

THEOREM 2.5. *The following are equivalent:*

1. $CV_0(T)$ *is a semi-Montel space;*
2. $CV_0(T)$ *is semireflexive;*
3. (i) T *is discrete, and*
   (ii) *if H is any bounded subset of $CV_0(T)$, then vH vanishes at infinity on T for every $v\in V$.*

Proof. Assume that $CV_0(T)$ is semireflexive. Fixing $t\in T$, let $\mathscr{U}$ denote a neighborhood base at t, and take $f\in CV_0(T)$ with $f(t) = 1$. Next, for each $U\in\mathscr{U}$, choose $\phi_U\in C_b(T)$ such that $0\leqslant\phi_U\leqslant 1$, $\phi_U(t) = 1$, and $\phi_U(T\backslash U) = 0$. Since $CV_0(T)$ is a module over $C_b(T)$, $H = \{\phi_U\colon U\in\mathscr{U}\}\subseteq CV_0(T)$. Moreover, since H is clearly bounded in $CV_0(T)$, H is weakly relatively compact. Now, it is obvious that the net $(\phi_U f)_U$ converges pointwise on T to the characteristic function $1_{\{t\}}$ of the singleton $\{t\}$, and thus, according to Theorem 2.2, $1_{\{t\}}\in CV_0(T)$ whereby $\{t\}$ is open in T; i.e., T is necessarily discrete. At this point, let us suppose that 3(ii) fails to hold. This being the case, there exists $v\in V$, an infinite sequence (t_k) in T, and a bounded sequence (f_k) in $CV_0(T)$ such that $v(t_k)|f_k(t_k)|\geqslant 1$ for each $k\in N$. Setting $h(t_k) = f_k(t_k)$, $k\in N$, and putting $h(t) = 0$ when $t\in T\backslash\{t_k\colon k\in N\}$, let h_k denote the (pointwise) product of h and the characteristic function of the set $\{t_j\colon j = 1,\ldots, k\}$ for $k = 1, 2,\ldots$ Since T is discrete and each h_k has compact support, (h_k) is a sequence in $CV_0(T)$. Further, this sequence is obviously bounded in $CV_0(T)$, as well as pointwise convergent on T to the function h. Another application of Theorem 2.2 thus yields that $h \in CV_0(T)$, which means that $F = \{t \in T: v(t)|h(t)| \geqslant 1\}$ must be compact. However, since $\{t_k: k \in N\} \subseteq F$, we have reached a contradiction, whereby 3 does indeed follow from 2. The fact that 3 implies 1 is an immediate consequence of [25, Theorem 2.1], and this certainly suffices to conclude the demonstration. ∎

At least when T is a k_R-space, Theorem 2.5 contains the above mentioned result due to Warner [33, p. 274] since condition 3(ii) of Theorem 2.5 is trivially satisfied in case $V = \mathscr{K}(T)$. Similarly, since the β_0-bounded subsets of $C_b(T)$ are uniformly bounded (cf. [31, p. 320]), 3(ii) is likewise readily seen to hold when $V = U(T)$ so that the corresponding result by Collins [7, p. 365] for the strict topology is also an immediate corollary of Theorem 2.5.

3. Weakly asymptotically almost periodic functions

Throughout the present section, we assume that X is a Banach space, while we write J_a to designate the subinterval $[a, \infty)$ of R corresponding to an arbitrary element $a \in R$. Further, let us recollect that a subset P of R is said to be *relatively dense* in R whenever there exists $l > 0$ such that $[t, t+l] \cap P \neq \emptyset$ for each $t \in R$; the relative density of a set P in a halfline J_a is defined in a completely analogous manner.

A function $f \in C(R, X)$ is termed *almost periodic* (a.p.) if, given any $\varepsilon > 0$, there exists a relatively dense set $P = P(\varepsilon)$ in R such that $\|f(t+\tau) - f(t)\| < \varepsilon$ for each $\tau \in P$ and every $t \in R$. Similarly, a function $f \in C(J_a, X)$ is said to be *asymptotically almost periodic* (a.a.p.) if, given $\varepsilon > 0$, there exist $M = M(\varepsilon) \geqslant a$ and a relatively dense set $P = P(\varepsilon)$ in J_M such that $\|f(t+\tau) - f(t)\| < \varepsilon$ for each $\tau \in P$ and all $t \in J_M$ with $t + \tau \geqslant M$. The set of all almost periodic members of $C(R, X)$ will be denoted by $AP(R, X)$, and we put $AAP(J_a, X) = \{f \in C(J_a, X): f \text{ is a.a.p.}\}$.

As is well known, if $f \in AP(R, X)$, then f is uniformly continuous and $f(R)$ is relatively compact in X (cf. [1]), which is equivalent to the assertion that the set $H(f) = \{f_\omega: \omega \in R\}$ of translates of f is relatively compact with respect to the compact-open topology $\varkappa$ on $C(R, X)$ (cf. [25]), but much more is true. Indeed, the classical characterization by S. Bochner [4] asserts that a function $f \in C(R, X)$ is a.p. if, and only if, $H(f)$ is a relatively compact subset of $C1_b(R, X)$, and therein lies the importance of the class $AP(R, X)$ with regard to questions of asymptotic behavior. We have recently shown [25, Theorem 3.1] that a function $f \in C(J_a, X)$ is a.a.p. if, and only if, the set $H^+(f) = \{f_\omega: \omega \in R^+\}$ of translates is a relatively compact subset of $C1_b(J_a, X)$.

Two weak forms of almost periodicity have been extensively considered in the literature. One notion (e.g., see [1], [21]) that has been used to advantage in the study of almost periodic equations in Banach spaces only enters into the picture in more than a formal way in the setting of vector valued functions with infinite dimensional range; we will say that a function $f\colon \mathbf{R}\to X$ is *weakly almost periodic* (w.a.p.) if $x'\circ f\in AP(\mathbf{R})$ for every $x'\in X'$. The second concept, on the other hand, has heretofore been studied almost exclusively in the scalar case. This alternative view, which derives from Bochner's criterion for almost periodicity, was introduced by Eberlein [10] in the context of scalar valued continuous functions on an abelian locally compact topological group, and it has since been considered in some detail in the more general framework of scalar functions on topological semigroups (e.g., see [6], [9]). However, there is no *a priori* reason for restricting attention to the scalar case; we will regard a function $f\in C(\mathbf{R}, X)$ as being *weakly almost periodic in the sense of Eberlein* (E.-w.a.p.) if $H(f)$ is a weakly relatively compact subset of $C1_{\mathrm{b}}(\mathbf{R}, X)$.

These two notions have the following obvious counterparts in the context of functions defined on a halfline:

(3.a) a function $f\colon J_a\to X$ is said to be *weakly asymptotically almost periodic* (w.a.a.p.) if $x'\circ f\in AAP(J_a)$ for every $x'\in X'$;

(3.b) a function $f\in C(J_a, X)$ is (as well) said to be *weakly almost periodic in the sense of Eberlein* (E.-w.a.p.) if $H^+(f)$ is a weakly relatively compact subset of $C1_{\mathrm{b}}(J_a, X)$.

Of course, Eberlein weak almost periodicity can also be placed in the context of vector valued continuous functions on topological semigroups. This view, albeit subject to the additional requirement that the functions in question have relatively compact range in the Banach space X, has been taken by S. Goldberg and P. Irwin [14] in an investigation which closely parallels that in the scalar case.

By definition, E.-w.a.p. functions enjoy a form of asymptotic behavior which lies in the same direction as that exhibited by a.p. and a.a.p. functions, but so also do w.a.p. and w.a.a.p. functions. Indeed, if we let X_w denote X under its associated weak topology $\sigma(X, X')$, then $f\colon \mathbf{R}\to X$ is w.a.p. if, and only if, $H(f)$ is a precompact subset of $C1_0(\mathbf{R}, X_w)$ (cf. [25, Theorem 3.3]); an analogous statement holds for w.a.a.p. functions.

As mentioned above, a.p. functions are necessarily uniformly continuous with relatively compact range, while the same is true for a.a.p. functions [25, Lemma 3.2]. Clearly, the range of an E.-w.a.p. function is weakly relatively compact in X. Also, it is known (cf. [6, p. 42]) that scalar valued E.-w.a.p. functions on locally compact groups (and thus on $\mathbf{R}$) are uniformly continuous, and we next note that this is likewise the case for E.-w.a.p. functions on a halfline.

LEMMA 3.1. *For $a\in\mathbf{R}$, if $f\in C(J_a)$ is weakly almost periodic in the sense of Eberlein, then f is uniformly continuous.*

Proof. Supposing that f is not uniformly continuous, there exist $\varepsilon > 0$ and sequences (s_n) and (t_n) in J_a such that $0 \leqslant t_n - s_n \leqslant 1/n$ and $|f(s_n) - f(t_n)| \geqslant \varepsilon$ for all $n \in \boldsymbol{N}$. As in the corresponding argument in [9] for functions on $\boldsymbol{R}$, let us consider $g_n = f_{t_n - a} - f_{s_n - a}$, $n \in \boldsymbol{N}$. Since (g_n) is a weakly relatively compact sequence in $C1_b(J_a)$, by going to a subsequence if necessary, we may assume that (g_n) is weakly convergent to some $g \in C_b(J_a)$. Consequently, $|g(a)| \geqslant \varepsilon$, and we can thus choose $\eta > 0$ so that $|\int_a^{a+\eta} g(u)\,du| > 0$. Now, for $h \in C_b(J_a)$, if we put $T_\eta(h) = \int_a^{a+\eta} h(u)\,du$, then $T_\eta \in C1_b(J_a)'$. Furthermore,

$$T_\eta(g) = \lim_n T_\eta(g_n) = \lim_n \Big[\int_{t_n}^{t_n+\eta} f(u)\,du - \int_{s_n}^{s_n+\eta} f(u)\,du\Big]$$

$$= \lim_n \Big[\int_{s_n+\eta}^{t_n+\eta} f(u)\,du - \int_{s_n}^{t_n} f(u)\,du\Big] = 0,$$

which provides the contradiction needed to complete the proof. ∎

The next result follows immediately from the foregoing lemma and the comments that preceded it.

THEOREM 3.2. *Given $a \in \boldsymbol{R}$, let $T \in \{J_a, \boldsymbol{R}\}$, and assume that X is a Banach space. If $f \in C(T, X)$ is weakly almost periodic in the sense of Eberlein, then f is weakly uniformly continuous* (i.e., $f\colon T \to X_w$ is uniformly continuous). *Moreover, weakly almost periodic functions from $\boldsymbol{R}$ into X and weakly asymptotically almost periodic functions from J_a into X are weakly uniformly continuous.*

Remark. We do not know whether such functions must as well be uniformly continuous as mappings into X. However, since the topology of X and the weak topology $\sigma(X, X')$ induce the same uniformity on absolutely convex compact subsets of the Banach space X (cf. [19, p. 386]), weak uniform continuity implies uniform continuity in the case of functions which happen to have relatively compact range in X. In particular, Theorem 3.2 immediately yields that an E.-w.a.p. function with relatively compact range is necessarily uniformly continuous (cf. [14, p. 11]).

Turning to sufficient conditions for weak almost periodicity in the sense of Eberlein which will be helpful to us in the sequel, we first observe that the study of vector valued E.-w.a.p. functions having relatively compact range actually reduces to a consideration of the scalar case. A version of the following result can also be found in [14], where it is stated without proof and attributed to M. Powell.

LEMMA 3.3. *Let $T = \boldsymbol{R}$ (respectively, $T = J_a$, where $a \in \boldsymbol{R}$), and assume that X is a Banach space. A function $f \in C_p(T, X)$ is then weakly almost periodic in the sense of Eberlein if, and only if, $x' \circ f$ is weakly almost periodic in the sense of Eberlein for every $x' \in X'$.*

Proof. Obviously, even if $f(T)$ were not relatively compact in X, if f is E.-w.a.p., then so also is $x' \circ f$ for each $x' \in X'$. For the converse, we would show that $H(f)$ (respectively, $H^+(f)$) is a weakly relatively compact subset of $Cl_b(T, X)$. Of course, since $f(T)$ is relatively compact in X, $H(f)$ (respectively, $H^+(f)$) is a bounded subset of $Cl_b(T, X)$. Moreover, since $x' \circ f$ is E.-w.a.p. for every $x' \in X'$, each of the functions $x' \circ f$ is uniformly continuous by Theorem 3.2. Again using the fact that $f(T)$ is relatively compact, this gives us that f is uniformly continuous whence, as previously noted, $H(f)$ (respectively, $H^+(f)$) is relatively compact in $C(T, X)$ with respect to the compact-open topology $\varkappa$. At this point, let us choose sequences (ω_n) in $\boldsymbol{R}$ (respectively, $\boldsymbol{R}^+$), (t_m) in T, and (x'_m) in B_1° such that, for some α, $\beta \in \boldsymbol{K}$,

$$\lim_m \lim_n \langle f_{\omega_n}(t_m), x'_m \rangle = \alpha \quad \text{and} \quad \lim_n \lim_m \langle f_{\omega_n}(t_m), x'_m \rangle = \beta;$$

the desired conclusion will then follow from Theorem 2.1 if we show that $\alpha = \beta$. To this end, we may assume that (f_{ω_n}) is $\varkappa$-convergent to a function $g \in C(T, X)$ whereby $g(T)$ is contained in the closure, call it K, of $f(T)$ in X. Since K is compact, we may further suppose that (x'_m) converges uniformly on K to some $x' \in B_1^\circ$. Thus, through an application of Theorem 2.1. based on the Eberlein weak almost periodicity of $x' \circ f$, we see that

$$\begin{aligned} \alpha &= \lim_m \langle g(t_m), x'_m \rangle = \lim_m \langle g(t_m), x' \rangle \\ &= \lim_m \lim_n \langle f_{\omega_n}(t_m), x' \rangle = \lim_n \lim_m \langle f_{\omega_n}(t_m), x' \rangle = \beta, \end{aligned}$$

and the proof is complete. ∎

To formulate the next result, we make use of a notion which derives from the standard concepts of α- and ω-limit sets for motions of dynamical systems: if $T \in \{J_a, \boldsymbol{R}\}$, $f \in C(T, X)$, and $\varkappa$ is the compact-open topology on $C(T, X)$, we let $\Lambda_\varkappa(f)$ denote the set of all $g \in C(T, X)$ for which there exists a sequence (ω_n) in $\boldsymbol{R}^+$ with $\omega_n \to +\infty$ or, in case $T = \boldsymbol{R}$, a sequence (ω_n) in $\boldsymbol{R}$ with $\omega_n \to -\infty$ such that (f_{ω_n}) is $\varkappa$-convergent to g. For future reference, moreover, if $f \in C_b(T, X)$, we will let $\Lambda_w(f)$ denote the corresponding collection of those $g \in C_b(T, X)$ for which there exists a sequence (ω_n) in $\boldsymbol{R}^+$ with $\omega_n \to +\infty$ or, in case $T = \boldsymbol{R}$, a sequence (ω_n) in $\boldsymbol{R}$ with $\omega_n \to -\infty$ such that (f_{ω_n}) is convergent to g in the weak topology induced on $C_b(T, X)$ by the Banach space $Cl_b(T, X)$. Obviously, given $f \in C_b(T, X)$, $\Lambda_\varkappa(f) \subseteq \Lambda_w(f)$ in case f is E.-w.a.p., while $\Lambda_\varkappa(f) = \Lambda_w(f)$ whenever f is a uniformly continuous E.-w.a.p. function with relatively compact range.

THEOREM 3.4. *Given $a \in \boldsymbol{R}$, let $T \in \{J_a, \boldsymbol{R}\}$, and assume that X is a Banach space. If $f \in C(T, X)$ is uniformly continuous and $f(T)$ is relatively compact in X, then f is weakly almost periodic in the sense of Eberlein whenever $\Lambda_\varkappa(f) \subseteq C_0(T, X) + \{\alpha_x\}$ for some $x \in X$, where $\alpha_x(t) = x$ for each $t \in T$.*

Proof. According to Lemma 3.3, it will suffice to verify that $x'\circ f$ is E.-w.a.p. for each $x'\in X'$, and so let us fix $x'\in X'$. For the case $T=\boldsymbol{R}$, the problem then reduces to showing that $H(x'\circ f)$ is a weakly relatively compact subset of $C1_b(\boldsymbol{R})$. Since $H(x'\circ f)$ is clearly bounded in $C1_b(\boldsymbol{R})$, however, this will follow from Theorem 2.1 if $H(x'\circ f)$ can be shown to satisfy the corresponding interchangeable double limits criterion. Thus, given sequences (ω_n) and (t_m) in $\boldsymbol{R}$ such that, for some $\alpha,\ \beta\in\boldsymbol{K}$,

$$\lim_m\lim_n(x'\circ f)_{\omega_n}(t_m)=\alpha \quad\text{and}\quad \lim_n\lim_m(x'\circ f)_{\omega_n}(t_m)=\beta,$$

we only need to show that $\alpha=\beta$ in order to complete the argument in the present instance. As one step in this direction, let us first recall that $H(f)$ is at least relatively compact in $C(\boldsymbol{R}, X)$ with respect to the compact-open topology $\varkappa$ since f is uniformly continuous and $f(\boldsymbol{R})$ is relatively compact in X. Now, if (ω_n) happens to be a bounded sequence, we may assume without loss of generality that $\omega_n\to\omega\in\boldsymbol{R}$. In this case, since we may further suppose that (f_{t_m}) is $\varkappa$-convergent to some $g\in C_b(\boldsymbol{R}, X)$, we deduce that

$$\alpha=\lim_m\langle f_{t_m}(\omega),\ x'\rangle=\langle g(\omega),\ x'\rangle=\lim_n\langle g(\omega_n),\ x'\rangle=\beta.$$

Similarly, we have that $\alpha=\beta$ in case (t_m) is a bounded sequence. Otherwise, if (ω_n) and (t_m) are both unbounded, we may assume that there exist $g,\ h\in C_0(\boldsymbol{R}, X)$ and $x\in X$ such that (f_{ω_n}) is $\varkappa$-convergent to $g+\alpha_x$ and (f_{t_m}) is $\varkappa$-convergent to $h+\alpha_x$, as well as that $\lim_n|\omega_n|=\lim_m|t_m|=+\infty$. Under these circumstances, we then have that

$$\alpha=\lim_m\langle g(t_m)+x,\ x'\rangle=\langle x,\ x'\rangle=\lim_n\langle h(\omega_n)+x,\ x'\rangle=\beta,$$

and this serves to conclude the proof in case $T=\boldsymbol{R}$. The argument for the case $T=J_a$ follows along similar lines. ∎

Remark. As one immediate consequence of the preceding result, we have that each $\phi\in C_0(\boldsymbol{R}, X)$ is E.-w.a.p. Therefore, given $g\in AP(\boldsymbol{R}, X)$ and $\phi\in C_0(\boldsymbol{R}, X), f=g+\phi$ is E.-w.a.p., but such a function f will be a.p. only when ϕ is identically zero on $\boldsymbol{R}$. In the direction of a converse, moreover, we point out that every E.-w.a.p. function has a related decomposition.

Indeed, from the general theory of weakly almost periodic semigroups of operators as developed by K. de Leeuw and I. Glicksberg [20, Section 4], it follows that each E.-w.a.p. function $f\in C_b(\boldsymbol{R}, X)$ can be uniquely represented as the sum of a function $g\in AP(\boldsymbol{R}, X)$ and a function $\phi\in W(\boldsymbol{R}, X)_0$, where we write $W(\boldsymbol{R}, X)_0$ to indicate the set of all E.-w.a.p. functions $\psi\in C_b(\boldsymbol{R}, X)$ such that $0\in\Lambda_w(\psi)$; an analogous decomposition holds as well for E.-w.a.p. functions on a halfline. Of course, there are classical precedents for such a result (cf. [6, p. 30]), and we also mention that an extension to the case of vector valued E.-w.a.p. functions has been obtained under special circumstances in [4]. With regard to applications, however, the lack of an "external" characterization of the members of $W(T, X)_0$ for $T \in \{\mathbf{R}, J_a\}$ is a definite shortcoming.

The remainder of this section will primarily be devoted to presenting examples which either further serve to distinguish between the types of almost periodicity under consideration or have import for our subsequent discussion of integrals. In this latter direction, we construct a class of examples of E.-w.a.p. functions in $C(\mathbf{R})$ which are not perturbations of a.p. functions by functions in $C_0(\mathbf{R})$, and thereby add to the extremely limited list of explicit instances that have previously been developed in this canonical setting (cf. [9] and [23]).

To set the stage for our first example, recall that if $f \in C(J_a, X)$ is E.-w.a.p., then $f(J_a)$ is weakly relatively compact in X and $x' \circ f$ is E.-w.a.p. for every $x' \in X'$. In contrast to Lemma 3.3, however, we next show that an assertion in the converse direction does not hold even when $x' \circ f$ vanishes at infinity on J_a (and thus belongs to $AP(J_a)$) for every $x' \in X'$.

Example 3.5. Consider the function $f\colon J_1 \to L^2[1, \infty)$ defined by

$$f(t)(s) = (\sqrt{t}/s)1_{J_t}(s)$$

for $s, t \in J_1 = [1, \infty)$. Then f is a uniformly continuous function with weakly relatively compact range which is weakly asymptotically almost periodic, but f is not weakly almost periodic in the sense of Eberlein.

Proof. As is obvious, $\|f(t)\|_2 = 1$ for every $t \in J_1$ whence $f(J_1)$ is weakly relatively compact in $L^2[1, \infty)$. A direct computation also shows that $\|f(t_1)-f(t_2)\|_2^2 \leqslant 2(\sqrt{t_2}-\sqrt{t_1})$ for $t_1, t_2 \in J_1$ with $t_1 \leqslant t_2$, and thus f is uniformly continuous. Next, for $h \in L^2[1, \infty)$ and $\varepsilon > 0$, choose $M \geqslant 1$ so that $\int_M^\infty |h(s)|^2\, ds < \varepsilon^2$. Then, for $t \in J_M$, we have that

$$|\langle f(t), h\rangle| = \Big|\int_1^\infty f(t)(s)\overline{h(s)}\, ds\Big| = \Big|\int_t^\infty (\sqrt{t}/s)\overline{h(s)}\, ds\Big|$$

$$\leqslant \|f(t)\|_2 \Big(\int_t^\infty |h(s)|^2\, ds\Big)^{1/2} < \varepsilon,$$

whereby the continuous map $\langle f(\cdot), h\rangle\colon J_1 \to \mathbf{K}$ vanishes at infinity on J_1 and therefore belongs to $AAP(J_1)$. In particular, this shows that f is w.a.a.p. To see that f is not E.-w.a.p., note that, for $m, n \in \mathbf{N}$, if $m \leqslant n$, then

$$\langle f_n(m), f(2m)\rangle = \int_1^\infty f(n+m)(s) f(2m)(s)\, ds = \sqrt{2m/(m+n)}$$

whence

$$\lim_n \langle f_n(m), f(2m)\rangle = 0 \qquad \text{for a fixed } m \in \mathbf{N},$$

whereas

$$\lim_m \langle f_n(m), f(2m)\rangle = \lim_m \sqrt{(m+n)/2m} = \sqrt{2}/2 \quad \text{for each } n\in N.$$

Since $(f(2m))$ is a sequence in the closed unit ball of $L^2[1, \infty)$, we can thus conclude from Theorem 2.1 that $H^+(f)$ fails to be weakly relatively compact in $C1_b(J_1, L^2[1, \infty))$.

Remark 3.6. 1. An example to the same end as 3.5 was considered in [14]. However, as P. Milnes [MR 80e:43010] has noted in his review, the example presented in [14] does not work since the given function is not bounded.

2. In particular, Example 3.5 provides a negative response to the question concerning a link between w.a.a.p. and E.-w.a.p. functions that had been raised in [25, Section 3].

As noted earlier, concrete instances of E.-w.a.p. functions which do not also satisfy yet stronger periodicity conditions are not exactly plentiful even in the scalar case. We bring the present section to an end by presenting examples of functions in this category which will help illustrate our results on integrals in the following section.

EXAMPLE 3.7. Let $I = (-1,1)$, fix $\gamma\in C_0(I)$, and consider $\varrho(\gamma)\colon \boldsymbol{R}^+\to \boldsymbol{K}$ defined by

$$\varrho(\gamma)(t) = \begin{cases} \gamma(t-2^k), & t\in(2^k-1, 2^k+1) \text{ for } k\in N,\\ 0, & \text{otherwise.}\end{cases}$$

(i) The function $\varrho(\gamma)$ is weakly almost periodic in the sense of Eberlein. However, $\varrho(\gamma)$ is asymptotically almost periodic only if $\gamma(t) = 0$ for every $t\in I$.

(ii) The function $\sigma(\gamma)\colon \boldsymbol{R}\to\boldsymbol{K}$ defined by

$$\sigma(\gamma)(t) = \begin{cases} \varrho(\gamma)(t), & t\in \boldsymbol{R}^+,\\ 0, & t\in \boldsymbol{R}\backslash \boldsymbol{R}^+,\end{cases}$$

is weakly almost periodic in the sense of Eberlein, but $\sigma(\gamma)$ is not almost periodic unless $\gamma(t) = 0$ for all $t\in I$.

Proof. In view of Theorem 3.4, since $\varrho(\gamma)$ is clearly bounded and uniformly continuous, we need only verify that $\Lambda_\varkappa(\varrho(\gamma))\subseteq C_0(\boldsymbol{R}^+)$ in order to conclude that $\varrho(\gamma)$ is E.-w.a.p., and so let us assume that (ω_n) is an increasing sequence in $\boldsymbol{R}^+$ with $\omega_n\to+\infty$ such that $(\varrho(\gamma)_{\omega_n})$ is $\varkappa$-convergent to a function $g\in C_b(\boldsymbol{R}^+)$. Further, suppose that $g(t)\neq 0$ for some $t\in\boldsymbol{R}^+$. In this case, there exists $n_t\in N$ such that $|\varrho(\gamma)(t+\omega_n)|\geqslant |g(t)|/2$ for all $n\in N$ with $n\geqslant n_t$, and hence there exist $\delta\in(0, 1)$ and an increasing sequence (k_n) in N such that $k_n\to+\infty$ and $t+\omega_n\in[2^{k_n}-\delta, 2^{k_n}+\delta]$ for all $n\in N$ with $n\geqslant n_t$. Fixing $\alpha\in[2, \infty)$, we put $u = t+\alpha$ and suppose that $g(u) > 0$. As before, there exist $n_u\in N$, $\eta\in(0,1)$, and an increasing sequence (j_n) in N such that $j_n\to+\infty$ and, for each $n\in N$ with

$n \geqslant n_u$, $u+\omega_n \in [2^{j_n}-\eta,\ 2^{j_n}+\eta]$. Now, choose $n \in N$ so that $n \geqslant \max\{n_t, n_u\}$ and $2^{k_n} > \alpha+2$. From the facts just noted, the following pair of relationships must then hold:

$$(3.c)\qquad \begin{aligned} 2^{k_n}-\delta-\omega_n+\alpha \leqslant u \leqslant 2^{k_n}+\delta-\omega_n+\alpha,\\ 2^{j_n}-\eta-\omega_n \leqslant u \leqslant 2^{j_n}+\eta-\omega_n. \end{aligned}$$

Suppose that $k_n < j_n$. From (3.c), we would then have that

$$\begin{aligned} 0 \leqslant 2^{k_n}+\delta-\omega_n+\alpha-(2^{j_n}-\eta-\omega_n) &= 2^{k_n}(1-2^{j_n-k_n})+\delta+\eta+\alpha\\ &\leqslant \alpha+(\delta+\eta)-2^{k_n}, \end{aligned}$$

which is impossible since $2^{k_n} > \alpha+2$. Thus $j_n \leqslant k_n$, in which case we conclude from (3.c) that

$$\begin{aligned} 0 \leqslant 2^{j_n}+\eta-\omega_n-(2^{k_n}-\delta-\omega_n+\alpha) &= 2^{j_n}(1-2^{k_n-j_n})+\eta+\delta-\alpha\\ &\leqslant (\eta+\delta)-\alpha. \end{aligned}$$

But this contradicts our choice of α, and so we have shown that $g(s) = 0$ for all $s \in [t+2, \infty)$; indeed, we can even conclude that there exists $a \in \boldsymbol{R}^+$ such that the support of g is contained in $[a, a+2]$.

Suppose, on the other hand, that $\varrho(\gamma)$ is a.a.p. and $\gamma(s) \neq 0$ for some $s \in (-1,1)$. Then, given $\varepsilon = |\gamma(s)|/2$, there exist $M \geqslant 0$ and a relatively dense set P in J_M such that $|\varrho(\gamma)(t+\tau)-\varrho(\gamma)(t)| < \varepsilon$ for each $\tau \in P$ and all $t \in J_M$; we let $l > 0$ be a measure of the density of P in J_M. Taking $n \in N$ so that $2^n \geqslant \max\{M+1, l+3\}$, we choose $\tau \in [2^n+2, 2^n+2+l] \cap P$, and put $t = s+2^n$. For this choice, however, $\varrho(\gamma)(t) = \gamma(s)$ and $t \in J_M$, while $\varrho(\gamma)(t+\tau) = 0$ since

$$2^{n+1}+1 \leqslant t+\tau \leqslant 2^{n+1}+l+3 \leqslant (2^2-1)2^n \leqslant 2^{n+2}-1,$$

and this contradiction completes the final step in our argument to establish 3.7(i).

Turning to 3.7(ii), since $\sigma(\gamma)$ is obviously not a.p. unless $\gamma(t) = 0$ for each $t \in (-1,1)$, we need only show that $\sigma(\gamma)$ is E.-w.a.p., and here, again invoking Theorem 3.4, it will suffice to show that $\Lambda_k(\sigma(\gamma)) \subseteq C_0(\boldsymbol{R})$. To this end, fix $g \in \Lambda_\varkappa(\sigma(\gamma))$, let (ω_n) be a sequence in $\boldsymbol{R}$ such that $(\sigma(\gamma)_{\omega_n})$ is $\varkappa$-convergent to g, and suppose that there exist $s, t \in \boldsymbol{R}$ with $s \leqslant t$ such that $g(s) \neq 0$ and $g(t) \neq 0$. Since $\lim_n \sigma(\gamma)(t+\omega_n) = g(t) \neq 0$, we may assume that (ω_n+s) is a sequence in $\boldsymbol{R}^+$ with $\omega_n+s \to +\infty$. Moreover, because $H^+(\varrho(\gamma))$ is relatively compact in $(C(\boldsymbol{R}^+), \varkappa)$ (as can readily be seen from [25, Corollary 2.5.1], for instance), there exists a subsequence (ω_{n_k}) of (ω_n) such that $(\varrho(\gamma)_{\omega_{n_k}+s})$ is $\varkappa$-convergent to a function $h \in \Lambda_\varkappa(\varrho(\gamma))$. This being the case,

$$g(s) = \lim_n \sigma(\gamma)_{\omega_n}(s) = \lim_n \varrho(\gamma)_{\omega_n+s}(0) = h(0),$$

while

$$g(t) = \lim \sigma(\gamma)_{\omega_n}(t) = \lim \varrho(\gamma)_{\omega_n+s}(t-s) = h(t-s).$$

As demonstrated above, however, the support of h is contained in an interval of length at most two, and so we see that $0 \leqslant t-s < 2$. Since it is now apparent that the support of g must also be contained in an interval of length two, the proof is complete. ∎

For the function $\varrho\colon C_0(I) \to C_b(\boldsymbol{R}^+)$ as described in Example 3.7, unless $\gamma \in C_0(I)$ is chosen so that $\int_{-1}^{1} \gamma(u)\,du = 0$, the corresponding indefinite integral $P(\gamma)\colon \boldsymbol{R}^+ \to \boldsymbol{K}$ defined by

$$P(\gamma)(t) = \int_0^t \varrho(\gamma)(u)\,du, \qquad t \in \boldsymbol{R}^+,$$

is not bounded. However, as we next note, ϱ can be modified so as to remedy this defect without affecting other pertinent properties.

EXAMPLE 3.8. Let $I = (-1,1)$, fix $\gamma \in C_0(I)$, and define $\varrho_e(\gamma)\colon \boldsymbol{R}^+ \to \boldsymbol{K}$ by

$$\varrho_e(\gamma)(t) = \begin{cases} \gamma(t-2^{2k}), & t \in (2^{2k}-1, 2^{2k}+1) \text{ for } k \in \boldsymbol{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Further, set $\varrho_0(\gamma) = \varrho(\gamma) - \varrho_e(\gamma)$, where $\varrho\colon C_0(I) \to C_b(\boldsymbol{R}^+)$ is the function defined in Example 3.7.

(i) The function $\alpha(\gamma) = \varrho_0(\gamma) - \varrho_e(\gamma)$ on $\boldsymbol{R}^+$ is weakly almost periodic in the sense of Eberlein, but $\alpha(\gamma)$ is asymptotically almost periodic only if $\gamma(t) = 0$ for every $t \in I$.

(ii) The function $\beta(\gamma)\colon \boldsymbol{R} \to \boldsymbol{K}$ defined by

$$\beta(\gamma)(t) = \begin{cases} \alpha(\gamma)(t), & t \in \boldsymbol{R}^+, \\ 0, & t \in \boldsymbol{R} \setminus \boldsymbol{R}^+, \end{cases}$$

is weakly almost periodic in the sense of Eberlein, but $\beta(\gamma)$ is not almost periodic unless $\gamma(t) = 0$ for all $t \in I$.

Proof. Since $|\alpha(\gamma)| = |\varrho_0(\gamma)| + |\varrho_e(\gamma)| = \varrho(|\gamma|)$, $\alpha(\gamma)$ is clearly not a.a.p. when $\varrho(\gamma)$ is not a.a.p. Further, since each $g \in \Lambda_\varkappa(\varrho(\gamma))$ has support contained in an interval of length two and $H^+(\varrho(\gamma))$ is a relatively compact subset of $(C(\boldsymbol{R}^+), \varkappa)$, the fact that $|\alpha(\gamma)| = \varrho(|\gamma|)$ immediately allows us to conclude that each $g \in \Lambda_\varkappa(\alpha(\gamma))$ also has support contained in an interval of length two, and so Theorem 3.4 applies to show that $\alpha(\gamma)$ is E.-w.a.p. Except for using $\sigma(\gamma)$ of Example 3.7 in the place of $\varrho(\gamma)$, the same argument will serve to establish (ii).

4. Integrals of asymptotically almost periodic functions

One striking indication of the difference between a.p. functions on $\boldsymbol{R}$ and their counterparts on a halfline lies in the failure of the Bohl–Bohr theorem to carry over to $AAP(J_a)$. Indeed, as the following example shows, bounded integrals of a.a.p. functions can even fail to be E.-w.a.p.

EXAMPLE 4.1. Consider the function $f: J_1 \to \boldsymbol{R}$ defined by

$$f(t) = \frac{1}{t}\cos(\log t), \qquad t \in J_1.$$

Then $f \in C_0(J_1)$ (whereby $f \in AAP(J_1)$) and the corresponding indefinite integral

$$F(t) = \int_1^t f(u)\,du = \sin(\log t), \qquad t \in J_1,$$

defines a bounded function on J_1, but F is not weakly almost periodic in the sense of Eberlein.

Proof. We need only verify that F is not E.-w.a.p., and so, for $m, n \in N$, let us put $t_m = \exp(m\pi)$ and $\omega_n = \exp\{\frac{1}{2}(4n-3)\pi\}$. Then

$$\begin{aligned} F(t_m+\omega_n) &= \sin\big(\log\big[\exp(m\pi)(1+\exp\{[\tfrac{1}{2}(4n-3)-m]\pi\})\big]\big) \\ &= \cos(m\pi)\sin\big(\log\big[1+\exp\{[\tfrac{1}{2}(4n-3)-m]\pi\}\big]\big) \end{aligned}$$

whence $\lim_n \lim_m F_{\omega_n}(t_m) = 0$. On the other hand, since we also have that

$$F(t_m+\omega_n) = \sin\big(\tfrac{1}{2}(4n-3)\pi\big)\cos\big(\log\big[1+\exp\{[m-\tfrac{1}{2}(4n-3)]\pi\}\big]\big),$$

$\lim_m \lim_n F_{\omega_n}(t_m) = 1$, whereby the interchangeable double limits condition does not hold. In view of Theorem 2.1, the proof is therefore complete. ∎

The function f defined in the preceding example is obviously not (improperly) Riemann integrable on J_1 (which is the same as to say that $\lim_{t\to+\infty}\int_1^t f(u)\,du$ does not exist). As one direct step toward resolving (0.b), we next show that this is precisely why the indefinite integral of f fails to be E.-w.a.p. In the process, we reveal an unexpected connection between the notions of weak almost periodicity in the sense of Eberlein and asymptotic almost periodicity.

THEOREM 4.2. *Given $a \in \boldsymbol{R}$ and a Banach space X, let $\phi \in C_0(J_a, X)$, and define $\Phi: J_a \to X$ by*

$$\Phi(t) = \int_a^t \phi(u)\,du \qquad \textit{for } t \in J_a.$$

Then the following are equivalent:

1. *Φ is asymptotically almost periodic;*
2. *Φ is weakly almost periodic in the sense of Eberlein;*
3. *ϕ is* (improperly) *Riemann integrable on J_a.*

Proof. Assume, first of all, that Φ is E.-w.a.p. Choosing a sequence (ω_n) in $\boldsymbol{R}^+$ with $\omega_n \to +\infty$, since $\Phi(J_a)$ is weakly relatively compact in X, the Eberlein–Smulian theorem allows us to assume that $(\Phi(a+\omega_n))$ is

$\sigma(X, X')$-convergent to some $x \in X$. Suppose, at this point, that there exist $\varepsilon > 0$ and a sequence (t_m) in J_a such that $t_m \to +\infty$ and $\|x - \Phi(t_m)\| \geqslant \varepsilon$ for each $m \in N$. This being the case, we can choose $x'_m \in B_1^\circ$ for each $m \in N$ so that $|\langle x - \Phi(t_m), x'_m \rangle| \geqslant \varepsilon$, and there is no loss of generality in assuming that there exist $\alpha, \beta \in K$ such that

$$\lim_m \langle x, x'_m \rangle = \alpha \quad \text{and} \quad \lim_m \langle \Phi(t_m), x'_m \rangle = \beta.$$

Now,

$$\lim_m \lim_n \langle \Phi_{\omega_n}(t_m), x'_m \rangle = \lim_m \lim_n \langle \Phi(a + \omega_n) + \int_a^{t_m} \phi_{\omega_n}(u)\, du, x'_m \rangle$$
$$= \lim_m \langle x, x'_m \rangle = \alpha,$$

while

$$\lim_n \lim_m \langle \Phi_{\omega_n}(t_m), x'_m \rangle = \lim_n \lim_m \langle \Phi(t_m) + \int_a^{a+\omega_n} \phi_{t_m - a}(u)\, du, x'_m \rangle = \beta,$$

from which, through an appeal to Theorem 2.1, we conclude that $\alpha = \beta$. However, taking $m \in N$ so that $|\alpha - \langle x, x'_m \rangle| < \varepsilon/2$ and $|\beta - \langle \Phi(t_m), x'_m \rangle| < \varepsilon/2$, we then have

$$|\langle x - \Phi(t_m), x'_m \rangle| \leqslant |\langle x, x'_m \rangle - \alpha| + |\alpha - \beta| + |\beta - \langle \Phi(t_m), x'_m \rangle| < \varepsilon,$$

which contradicts our choice of x'_m and thereby shows that $\lim_{t \to +\infty} \int_a^t \phi(u)\, du = x$; i.e., ϕ is necessarily Riemann integrable on J_a. This shows that 2 implies 3. All other implications in Theorem 4.2 are straightforward. ∎

Obviously, if $\Phi \in C_b(J_a, X)$ and Φ is Riemann integrable on J_a, the corresponding indefinite integral must be a.a.p. As a sufficient condition for the integral of an arbitrary function $f \in AAP(J_a, X)$ to also be a.a.p., however, Riemann integrability of f on J_a is clearly much too restrictive since a uniformly continuous Riemann integrable function on J_a with values in a Banach space must necessarily vanish at infinity on J_a. Nonetheless, it will shortly become evident that Theorem 4.2 does have an important bearing on the solution of (0.b).

Before stating our main result, we pause to recall the following basic fact from [25, Theorem 3.4] concerning the decomposition of a.a.p. functions: If X is a Banach space and $f \in AAP(J_a, X)$, then there are unique functions $g \in AP(R, X)$ and $\phi \in C_0(J_a, X)$ such that

$$f = g|J_a + \phi; \tag{4.a}$$

we will refer to ϕ as the *critical part* of f.

THEOREM 4.3. *Given* $a \in \boldsymbol{R}$ *and a Banach space* X, *fix* $f \in AAP(J_a, X)$, *and define* $F\colon J_a \to X$ *by*

$$F(t) = \int_a^t f(u)\,du \quad \text{for } t \in J_a.$$

In case either

(4.b) $F(J_a)$ *is bounded in* X *and* X *does not contain an isomorphic copy of* c_0,

or

(4.c) $\qquad F(J_a)$ *is weakly relatively compact in* X,

then the following are equivalent:

(i) F *is asymptotically almost periodic;*

(ii) F *is weakly almost periodic in the sense of Eberlein;*

(iii) *the critical part of* f *is* (improperly) *Riemann integrable on* J_a.

In our approach to proving 4.3, Theorem 4.2 gives us a hold on the critical part of an a.a.p. function. The following sequence of technical lemmas will allow us to utilize the work of Kadets [18] in treating the a.p. part of the decomposition (4.a). Moreover, the added generality built into the first of these lemmas will be helpful to us later in this section when the discussion turns to integrals of E.-w.a.p. functions.

LEMMA 4.4. *Given* $a \in \boldsymbol{R}$ *and a Banach space* X, *fix* $f \in C(J_a, X)$, *and let* $g \in AP(\boldsymbol{R}, X)$. *Setting*

$$F(t) = \int_a^t f(u)\,du \quad \text{and} \quad G_a(t) = \int_a^t g(u)\,du \quad \text{for } t \in J_a,$$

if $\phi = f - g|J_a$ *is weakly almost periodic in the sense of Eberlein and* $0 \in \Lambda_w(\phi)$, *then* $G_a(J_a) \subseteq 2\overline{ac}(F(J_a))$.

Proof. Certainly, $G_a(a) = 0$ belongs to the closed absolutely convex hull $\overline{ac}(F(J_a))$ of $F(J_a)$, and so let us consider $t \in J_a$ with $t > a$. Now, let $\varepsilon > 0$, fix a finite set $\{x_i'\colon i = 1, \ldots, m\}$ in X', and put $\alpha = \max\{\|x_i'\|\colon i \in \{1, \ldots, m\}\} + 1$. For $\eta = \varepsilon/(2(t-a)\alpha)$, we can then choose a relatively dense subset P of $\boldsymbol{R}$ with density constant $l > 0$ such that $\|g(u+\tau) - g(u)\| < \eta$ for every $u \in \boldsymbol{R}$ and all $\tau \in P$. At this point, we would show that there exists $\omega \in \boldsymbol{R}^+$ so that $|\langle \phi_\omega(s), x_i' \rangle| < \eta$ for each $i \in \{1, \ldots, m\}$ and every $s \in [a, t+l]$. Since ϕ is E.-w.a.p., however, ϕ is weakly uniformly continuous by Theorem 3.2 and the range of ϕ must as well be weakly relatively compact in X. Thus, as readily follows from [25, Corollary 2.5.1], $H^+(\phi)$ is relatively compact in $C(J_a, X_w)$ with respect to the compact-open topology. Moreover, since $0 \in \Lambda_w(\phi)$, there exists a sequence (ω_n) in $\boldsymbol{R}^+$ (with $\omega_n \to +\infty$) such that (ϕ_{ω_n}) converges weakly to zero in $Cl_b(J_a, X)$, and we need only note that some subnet of (ϕ_{ω_n}) will therefore converge to zero in $(C(J_a, X_w), \varkappa)$ in order to establish our claim and thereby obtain $\omega \in \boldsymbol{R}^+$ with the desired property. Consequently, choosing $\tau \in P \cap [\omega, \omega + l]$ and putting $\sigma = \tau - \omega$, whence $\sigma \in [0, l]$, if $i \in \{1, \ldots, m\}$, we have that

$$\begin{aligned}
&|\langle G_a(t) - (F(t+\tau) - F(a+\tau)), x_i' \rangle| \\
&\quad \leqslant |\langle G_a(t) - \int_a^t g(u+\tau)\,du, x_i' \rangle| + |\langle \int_a^t g(u+\tau)\,du - \int_{a+\tau}^{t+\tau} f(u)\,du, x_i' \rangle| \\
&\quad \leqslant \|x_i'\| \int_a^t \|g(u) - g(u+\tau)\|\,du + |\langle \int_a^t \phi(u+\tau)\,du, x_i' \rangle| \\
&\quad \leqslant \|x_i'\|(t-a)\eta + \int_a^t |\langle \phi_\omega(u+\sigma), x_i' \rangle|\,du < \varepsilon,
\end{aligned}$$

which is to say that $G_a(t)$ belongs to the weak closure of $2ac(F(J_a))$. For a convex set, however, the weak closure coincides with the closure in X, and so the proof is complete. ∎

LEMMA 4.5. *Given $a \in \boldsymbol{R}$ and a Banach space X, let $g \in AP(\boldsymbol{R}, X)$, put*

$$G_a(t) = \int_a^t g(u)\,du \quad \textit{for } t \in J_a,$$

and set

$$G(t) = \begin{cases} \int_0^t g(u)\,du, & t \in \boldsymbol{R}^+, \\ -\int_t^0 g(u)\,du, & t \in \boldsymbol{R} \setminus \boldsymbol{R}^+ \end{cases}$$

Then $G(\boldsymbol{R}) \subseteq G([0, |a|]) + 2\overline{ac}(G_a(J_a))$.

Proof. Fixing $t \in \boldsymbol{R}$, first assume that $t > |a|$. If $a \geqslant 0$, then $G(t) = G(a) + G_a(t)$, while $G(t) = G_a(t) - G_a(0)$ in case $a < 0$. Consequently, we may suppose that $t < 0$. For this case, given $\varepsilon > 0$, take P to be a relatively dense subset of $\boldsymbol{R}$ such that $\|g(u+\tau) - g(u)\| < \varepsilon/|t|$ for each $u \in \boldsymbol{R}$ and all $\tau \in P$. Choosing $\tau \in P$ so that $t + \tau \geqslant a$, we then have

$$\begin{aligned}
\|G(t) - (G_a(t+\tau) - G_a(\tau))\| &= \|G(t) + \int_{t+\tau}^{\tau} g(u)\,du\| \\
&= \|\int_t^0 g(u+\tau)\,du - \int_t^0 g(u)\,du\| \\
&\leqslant \int_t^0 \|g(u+\tau) - g(u)\|\,du < \varepsilon,
\end{aligned}$$

which completes the proof. ∎

Taken together, Lemmas 4.4 and 4.5 yield the following result.

LEMMA 4.6. *Given $a \in \boldsymbol{R}$ and a Banach space X, fix $f \in C(J_a, X)$, let $g \in AP(\boldsymbol{R}, X)$, and assume that $f-g|J_a \in C_0(J_a, X)$. Setting*

$$F(t) = \int_a^t f(u)\,du \quad \text{for } t \in J_a \qquad \text{and} \qquad G(t) = \int_a^t g(u)\,du \quad \text{for } t \in \boldsymbol{R},$$

if $F(J_a)$ is bounded, or weakly relatively compact, or relatively compact in X, then the same is true for $G(\boldsymbol{R})$.

Proof of Theorem 4.3. Since $f \in AAP(J_a, X)$, there are unique functions $g \in AP(\boldsymbol{R}, X)$ and critical part $\phi \in C_0(J_a, X)$ such that $f = g|J_a + \phi$ [25, Theorem 3.4]; we put $\Phi(t) = \int_a^t \phi(u)\,du$, $t \in J_a$, and define the indefinite integrals G_a: $J_a \to X$ and G: $\boldsymbol{R} \to X$ of g as in Lemma 4.5. At this point, assume that F is E.-w.a.p. Since $F(J_a)$ is then weakly relatively compact in X, $G(\boldsymbol{R})$ is weakly relatively compact in X by Lemma 4.6. According to [18, Theorem 2], this means that $G \in AP(\boldsymbol{R}, X)$ whence $G_a = G|J_a - G(a) \in AAP(J_a, X)$. Consequently, $\Phi = F - G_a$ is E.-w.a.p., and so Theorem 4.2 applies to give us that ϕ is Riemann integrable on J_a. Suppose, on the other hand, that the critical part ϕ of f is Riemann integrable on J_a. As follows from another application of Theorem 4.2, Φ is then a.a.p. Moreover, since either (4.b) or (4.c) is satisfied, Lemma 4.6 shows that either $G(\boldsymbol{R})$ is weakly relatively compact in X or, in case $c_0 \not\subseteq X$, $G(\boldsymbol{R})$ is at least bounded. Again by [18, Theorem 2] in the first instance, or from [18, Theorem 1] in the second, G must necessarily be a.p. in either event. Hence, G_a is a.a.p., and therefore so also is $F = G_a + \Phi$. Since (i) clearly implies (ii), the proof is now complete. ∎

Remark. 1. The surprising aspect of Theorem 4.3, to us, is the equivalence between propositions (i) and (ii). It is the proof of this equivalence that took all the preparation on weak compactness and E.-w.a.p. functions in the preceding sections. At this point, one might wonder about the practical importance of the concept of E.-w.a.p. functions in general. In a forthcoming publication [28], we shall show that functions of this type occur naturally in the context of the abstract Cauchy problem: Assuming that a linear operator A: $D(A) \subset X \to X$ on a Banach space X is the infinitesimal generator of a uniformly bounded C_0-semigroup $(S(t))_{t \geqslant 0}$ of continuous linear operators on X, consider the abstract Cauchy problem

$$\text{(CP)} \qquad \begin{aligned} \dot{x}(t) &= Ax(t), \qquad t \in \boldsymbol{R}^+, \\ x(0) &= x_0 \end{aligned}$$

associated with A. We show in [28] that, for $x_0 \in D(A)$, the (unique) strong solution $x(t) = S(t)x_0$ of (CP) is E.-w.a.p. provided it has weakly relatively compact range. In particular, whenever, in addition to the above assumptions, X is a reflexive Banach space, all solutions to (CP) are necessarily E.-w.a.p. Teamed with the Jacobs–De Leeuw–Glicksberg theory of weakly almost periodic semigroups of operators, this automatically implies the existence of classically *almost periodic* solutions to (CP). For details and further results on the concept of E.-w.a.p. motions of semigroups of operators, we refer to [28].

2. Since functions encountered in practice tend to be integrals, the equivalence of (i) and (ii) in Theorem 4.3 provides some insight into the difficulty (even in the scalar case) of finding examples of functions on a halfline for which the set of translates forms a weakly relatively compact set that is not already relatively compact in the topology of uniform convergence. As we point out in the following example, however, this equivalence can fail if the integrand only happens to be E.-w.a.p. instead of a.a.p.

EXAMPLE 4.7. Let $I=(-1,1)$, and choose $\gamma\in C_0(I)$ such that γ is not identically zero on I but $\int_{-1}^{1}\gamma(u)\,du=0$. Now, consider the corresponding Eberlein weakly almost periodic function $\varrho(\gamma)$ from Example 3.7, and put $P(\gamma)(t)=\int_0^t\varrho(\gamma)(u)\,du$ for each $t\in\boldsymbol{R}^+$ The indefinite integral $P(\gamma)$ is then weakly almost periodic in the sense of Eberlein on $\boldsymbol{R}^+$, but $P(\gamma)$ is not asymptotically almost periodic.

Proof. Setting $\Gamma(t)=\int_{-1}^{t}\gamma(u)\,du$ for $t\in I$, it suffices to note that $\Gamma\in C_0(I)$ and $\varrho(\Gamma)=P(\gamma)$; the requisite properties were established for $\varrho(\Gamma)$ in Example 3.7. ∎

We next turn our attention to integrals of w.a.a.p. functions. As our point of departure, we note that, under reasonable circumstances, a decomposition in the spirit of (4.a) can as well be obtained in this setting.

LEMMA 4.8. *Assume that X is a Banach space, and fix $a\in\boldsymbol{R}$. A function $f\colon J_a\to X$ with weakly relatively compact range is then weakly asymptotically almost periodic if, and only if, there is a unique weakly almost periodic function $g\colon\boldsymbol{R}\to X$ and a unique function $\phi\in C_0(J_a, X_w)$ such that*

$$f=g\,|\,J_a+\phi. \tag{4.d}$$

Proof. If the decomposition (4.d) holds for some $g\in AP(\boldsymbol{R}, X_w)$ and $\phi\in C_0(J_a, X_w)$, then f is clearly w.a.a.p. For the converse, assuming that $f\in AAP(J_a, X_w)$, we put $C=\overline{ac}(f(J_a))$, let X'^* denote the algebraic dual of X', set $Y=(X'^*, \sigma(X'^*, X'))$, and consider f as a mapping from J_a into Y. Since Y is complete and $f\in AAP(J_a, Y)$, [25, Theorem 3.4] asserts that there are unique functions $g\in AP(\boldsymbol{R}, Y)$ and $\phi\in C_0(J_a, Y)$ such that $f=g\,|\,J_a+\phi$. Fixing $t\in\boldsymbol{R}$, suppose that $g(t)\notin C$. Since C is $\sigma(X, X')$-compact, whence $\sigma(X'^*, X')$-compact, there then exists $x'\in X'$ such that $\langle g(t), x'\rangle>1$ and $|\langle x, x'\rangle|\leqslant 1$ for all $x\in C$. Now, setting $\varepsilon=\langle g(t), x'\rangle-1$, let P be a relatively dense subset of $\boldsymbol{R}$ such that $|\langle g(s+\tau)-g(s), x'\rangle|<\varepsilon/2$ for every $s\in\boldsymbol{R}$ and all $\tau\in P$. Further, choose $M\geqslant a$ so that $|\langle\phi(s), x'\rangle|<\varepsilon/2$ in case $s\in J_a$ with $s\geqslant M$. For $\tau\in P$ such that $t+\tau\geqslant M$, we would then have that

$$|\langle g(t), x'\rangle|\leqslant|\langle g(t)-g(t+\tau), x'\rangle|+|\langle f(t+\tau)-\phi(t+\tau), x'\rangle|<1+\varepsilon,$$

which is an obvious contradiction. Consequently, $g(\boldsymbol{R})\subseteq C$, whereby $g\in AP(\boldsymbol{R}, X_w)$ and $\phi\in C_0(J_a, X_w)$, and the proof is complete. ∎

In keeping with our terminology for a.a.p. functions taking values in a Banach space, if $f\colon J_a \to X$ is w.a.a.p. and has weakly relatively compact range, we will also refer to the unique function $\phi \in C_0(J_a, X_w)$ from the decomposition (4.d) as the *critical part* of f.

THEOREM 4.9. *Given $a \in \boldsymbol{R}$ and a Banach space X, assume that $f \in C(J_a, X)$ is a weakly asymptotically almost periodic function with weakly relatively compact range, and define $F\colon J_a \to X$ by*

$$F(t) = \int_a^t f(u)\,du \quad \text{for } t \in J_a.$$

Then F is weakly asymptotically almost periodic if, and only if, $F(J_a)$ is bounded in X and the critical part ϕ of f satisfies the following condition:

(4.e) *for each $x' \in X'$, $x' \circ \phi$ is* (improperly) *Riemann integrable on J_a.*

Proof. Given $x' \in X'$, we note that $x' \circ F(t) = \int_a^t x' \circ f(u)\,du$ for $t \in J_a$, $x' \circ f \in AAP(J_a)$, and $x' \circ \phi$ is the critical part of $x' \circ f$. The conclusion is now an immediate consequence of Theorem 4.3. ∎

Remark. In view of the classical Bohl–Bohr theorem, of course, if $f \in C(\boldsymbol{R}, X)$ is w.a.p. and $F(t) = \int_0^t f(u)\,du$ for $t \in \boldsymbol{R}$, then F is w.a.p. exactly when $F(\boldsymbol{R})$ is bounded in X (cf. [1, p. 59]).

We close this circle of ideas by giving a further curious example of an a.a.p. function with an integral that is not E.-w.a.p. Contrary to the situation in Example 4.1, however, the integral in this case is w.a.a.p.

EXAMPLE 4.10. Taking $X = Cl_0(J_1)$, consider the function $f\colon \boldsymbol{R}^+ \to X$ defined by

$$f(t)(s) = \sin\left(\frac{1}{t+s}\right) - t(t+s)^{-2}\cos\left(\frac{1}{t+s}\right)$$

for $t \in \boldsymbol{R}^+$ and $s \in J_1$. Then

(i) $f \in C_0(\boldsymbol{R}^+, X)$ so that, in particular, $f \in AAP(\boldsymbol{R}^+, X)$, but f is not Riemann integrable on $\boldsymbol{R}^+$

Moreover, setting

$$F(t) = \int_0^t f(u)\,du \quad \text{for } t \in \boldsymbol{R}^+,$$

(ii) $F \in C_b(\boldsymbol{R}^+, X)$, but $F(\boldsymbol{R}^+)$ is not weakly relatively compact in X;

(iii) F is weakly asymptotically almost periodic, but F is not weakly almost periodic in the sense of Eberlein.

Proof. As is straightforward to check, f is a well defined continuous function from $\boldsymbol{R}^+$ into X. Thus, since

$$\|f(t)\| \leqslant \sin\left(\frac{1}{t+1}\right) + \frac{t}{(t+1)^2} \quad \text{for each } t \in \boldsymbol{R}^+,\ f \in C_0(\boldsymbol{R}^+, X).$$

Next, observe that

$$F(t)(s) = t\sin\left(\frac{1}{t+s}\right) \quad \text{for all } t\in \boldsymbol{R}^+ \text{ and each } s\in J_1.$$

From this, we see that

$$\|F(t)\| = t\sin\left(\frac{1}{t+1}\right) \quad \text{for each } t\in \boldsymbol{R}^+,$$

and therefore $F(\boldsymbol{R}^+)$ is indeed a bounded subset of X. Be that as it may, since $\lim_{t\to+\infty} F(t)(s) = 1$ for each $s\in J_1$, f is obviously not Riemann integrable on $\boldsymbol{R}^+$ so that, as a consequence of Theorem 4.2, F is not E.-w.a.p. More to the point, however, a glance at Condition 3(iii) of Theorem 2.2 immediately shows that $F(\boldsymbol{R}^+)$ is not even weakly relatively compact in X. To see that F is w.a.a.p., first note that, given $\varepsilon > 0$ and $M\in J_1$, there exists $t_0\in \boldsymbol{R}^+$ such that $|F(t)(s)-1| < \varepsilon$ for all $s\in[1, M]$ whenever $t\in \boldsymbol{R}^+$ with $t \geqslant t_0$. Consequently, taking $\mu\in C1_0(J_1)'$, since μ is a bounded Radon measure on J_1, we have that

$$\lim_{t\to+\infty} \int_0^t \mu\circ f(u)\,du = \lim_{t\to+\infty} \left[\mu(J_1) + \int_{J_1} (F(t)-1)\,d\mu\right] = \mu(J_1);$$

i.e., $\mu\circ f$ is Riemann integrable on $\boldsymbol{R}^+$ An application of Theorem 4.9 now completes the argument. ∎

Some observations concerning integrals of E.-w.a.p. functions will bring the section to an end. Our first result, a version of Theorem 4.3 for functions defined on the entire real line, clarifies the extent to which a function in $AP(\boldsymbol{R}, X)$ can be perturbed by a member of $C_0(\boldsymbol{R}, X)$ and yet have an integral that is at least E.-w.a.p.

THEOREM 4.11. *Given a Banach space X, take $g\in AP(\boldsymbol{R}, X)$ and $\phi\in C_0(\boldsymbol{R}, X)$, put $f = g+\phi$, and set*

$$F(t) = \int_0^t f(u)\,du \quad \textit{for } t\in \boldsymbol{R}.$$

Then F is weakly almost periodic in the sense of Eberlein if, and only if, either

(i) *$F(\boldsymbol{R})$ is weakly relatively compact in X, or*

(ii) *$c_0 \nsubseteq X$ and $F(\boldsymbol{R})$ is bounded in X,*

and ϕ is (improperly) *Riemann integrable on $\boldsymbol{R}$ with*

$$\lim_{t\to+\infty} \int_0^t \phi(u)\,du = \lim_{t\to-\infty} \int_0^t \phi(u)\,du. \tag{4.f}$$

Proof. To begin, let us put $\Phi(t) = \int_0^t \phi(u)\,du$ for $t\in \boldsymbol{R}$. In either direction, since (i) certainly holds if F is E.-w.a.p., Lemma 4.6 can again be combined with the generalized Bohl–Bohr theorems due to Kadets [18] to show that $F-\Phi$ is a.p. At this point, suppose that F is E.-w.a.p. Setting $\theta(t) = -t$ for $t \in \boldsymbol{R}$, both $F|\boldsymbol{R}^+$ and $\phi o \theta|\boldsymbol{R}^+$ are therefore Riemann integrable on $\boldsymbol{R}^+$ by Theorem 4.3, we have that ϕ is Riemann integrable on $\boldsymbol{R}$. Moreover, as a consequence of our earlier observation that $F-\Phi$ is a.p., Φ is E.-w.a.p., and a straightforward application of the interchangeable double limits criterion (Theorem 2.1) now shows that (4.f) must also be satisfied. For the converse, in view of the integrability conditions on ϕ, we have from Theorem 3.4 that Φ is E.-w.a.p. Again using the fact that $F-\Phi$ is a.p., we thus reach the desired conclusion. ∎

From Example 4.7, we know that there do exist E.-w.a.p. functions on a halfline which are not a.a.p. and yet which have an integral that is E.-w.a.p. Similarly, choosing $\gamma \in C_0(I)$ as in Example 4.7 and using σ from Example 3.7 (ii) in place of ϱ, $\sigma(\gamma)$ is an E.-w.a.p. function on $\boldsymbol{R}$ which is not covered under Theorem 4.11, but the indefinite integral of $\sigma(\gamma)$ is nonetheless E.-w.a.p. An examination of the proofs for Examples 4.7 and 3.7 will show that, in each of these two instances, the integral is necessarily E.-w.a.p. because it satisfies the sufficient conditions given in Theorem 3.4, and the same is true for E.-w.a.p. integrals of functions in either $C_0(\boldsymbol{R}, X)$ or $C_0(J_a, X)$ as can be ascertained from the arguments for Theorems 4.11 and 4.2, respectively. As we next demonstrate, at least the limit set condition from 3.4 can also be necessary in the case of integrals.

THEOREM 4.12. *Assume that X is a Banach space, let $T = \boldsymbol{R}$ (respectively, $T = J_a$, where $a \in \boldsymbol{R}$), take $f \in C(T, X)$ to be a weakly almost periodic function in the sense of Eberlein such that $\Lambda_w(f) \subseteq C_0(T, X)$, and put*

$$F(t) = \int_0^t f(u)\,du \qquad \text{for } t \in \boldsymbol{R}$$

(respectively, $F(t) = \int_a^t f(u)\,du$ for $t \in J_a$). If F is weakly almost periodic in the sense of Eberlein, then $\Lambda_w(F) \subseteq C_0(T, X) + \{\alpha_x\}$ for some $x \in X$, where $\alpha_x(t) = x$ for each $t \in T$.

Proof. We consider the case $T = \boldsymbol{R}$; the argument will follow the same pattern when $T = J_a$. Now, given $G \in \Lambda_w(F)$, there exists a sequence (ω_n) in $\boldsymbol{R}$ with either $\omega_n \to +\infty$ or $\omega_n \to -\infty$ such that (F_{ω_n}) is weakly convergent to G in $C1_b(\boldsymbol{R}, X)$, and we may further assume that (f_{ω_n}) is weakly convergent to a function $\phi \in C_0(\boldsymbol{R}, X)$. Fixing $t \in \boldsymbol{R}$, since

$$F_{\omega_n}(t) = \int_0^{\omega_n} f(u)\,du + \int_{\omega_n}^{t+\omega_n} f(u)\,du = F_{\omega_n}(0) + \int_0^t f_{\omega_n}(u)\,du$$

for each $n \in N$, we then conclude that $G(t) = G(0) + \int_0^t \phi(u)\,du$. Thus, setting $\Phi(t) = \int_0^t \phi(u)\,du$ for each $t \in \boldsymbol{R}$, we have that $\Phi = G - G(0)$ is E.-w.a.p. since G has this property. By Theorem 4.11, therefore, ϕ is Riemann integrable on

$\boldsymbol{R}$ and there exists $z \in X$ such that $z = \lim_{t\to+\infty} \Phi(t) = \lim_{t\to-\infty} \Phi(t)$; we put $x = z+G(0)$. At this point, because $\|G(t)-x\| = \|\Phi(t)-z\|$ for each $t \in \boldsymbol{R}$, it is obvious that $G-\alpha_x \in C_0(\boldsymbol{R}, X)$. Suppose, an the other hand, that H is any other member of $\Lambda_w(F)$, let (t_m) be a sequence in $\boldsymbol{R}$ with either $t_m \to +\infty$ or $t_m \to -\infty$ such that (F_{t_m}) is weakly convergent to H in $C1_b(\boldsymbol{R}, X)$, and choose $y \in X$ so that $H-\alpha_y \in C_0(\boldsymbol{R}, X)$. Given $x' \in X'$, we would then have

$$\lim_m \lim_n \langle F_{\omega_n}(t_m), x'\rangle = \lim_m \langle G(t_m), x'\rangle = \langle x, x'\rangle,$$

while

$$\lim_n \lim_m \langle F_{\omega_n}(t_m), x'\rangle = \lim_n \langle H(\omega_n), x'\rangle = \langle y, x'\rangle.$$

Since F is E.-w.a.p., however, Theorem 2.1 then gives us that $\langle x, x'\rangle = \langle y, y'\rangle$, whereby $x = y$, and the proof for the case $T = \boldsymbol{R}$ is thus complete. ∎

COROLLARY 4.13. *Assume that X is a Banach space, and let $T = \boldsymbol{R}$ (respectively, $T = J_a$, where $a \in \boldsymbol{R}$). Further, taking $\phi \in C(T, X)$ to be a weakly almost periodic function in the sense of Eberlein such that $\Lambda_w(\phi) \subseteq C_0(T, X)$, fix $g \in AP(\boldsymbol{R}, X)$, and put $f = g+\phi$ (respectively, $f = g\,|\,J_a+\phi$). Setting*

$$F(t) = \int_0^t f(u)\,du \quad \text{and} \quad \Phi(t) = \int_0^t \phi(u)\,du \quad \text{for } t \in \boldsymbol{R}$$

(respectively, $F(t) = \int_a^t f(u)\,du$ and $\Phi(t) = \int_a^t \phi(u)\,du$ for $t \in J_a$), the following are then equivalent in case $F(T)$ is relatively compact in X:

(i) *F is weakly almost periodic in the sense of Eberlein;*

(ii) *$\Lambda_w(\Phi) \subseteq C_0(T, X)+\{\alpha_x\}$ for some $x \in X$, where $\alpha_x(t) = x$ for each $t \in T$.*

Proof. To begin, let us set $G(t) = \int_0^t g(u)\,du$ for $t \in \boldsymbol{R}$. Now, whether $T = \boldsymbol{R}$ or $T = J_a$, if ψ belongs to the weak closure $\mathrm{cl}_w(H^+(\phi))$ of $H^+(\phi)$ in $C1_b(T, X)$, then $H^+(\psi) \subseteq \mathrm{cl}_w(H^+(\phi))$. Thus, since $\Lambda_w(\phi) \subseteq C_0(T, X)$, we see that $0 \in \mathrm{cl}_w(H^+(\phi))$. From this, it then readily follows that $0 \in \Lambda_w(\phi)$, and we can even conclude that $0 \in \Lambda_w(\phi\,|\,\boldsymbol{R}^+)$ in case $T = \boldsymbol{R}$. Since Lemma 4.4 therefore applies in the present setting, this result together with Lemma 4.5 gives us that the range of G is relatively compact in X whereby $G \in AP(\boldsymbol{R}, X)$ (cf. [1, p. 53]). In turn, $\Phi = F-G$ must as well have relatively compact range in X, and Φ, moreover, will be E.-w.a.p. if, and only if, F is E.-w.a.p. Hence, we need only apply Theorem 3.4 to see that (ii) implies (i), while the converse follows as an immediate consequence of Theorem 4.12. ∎

Our concluding result shows that the necessary condition of Theorem 4.12 is not automatic.

EXAMPLE 4.14. For $I = (-1, 1)$, choose $\gamma \in C_0(I)$ so that $\int_{-1}^{1} \gamma(u)\,du \neq 0$, and consider the corresponding functions $\alpha(\gamma)$ on $\boldsymbol{R}^+$ and $\beta(\gamma)$ on $\boldsymbol{R}$ from Example 3.8. Then $\alpha(\gamma)$ and $\beta(\gamma)$ are uniformly continuous (indeed, Eberlein weakly almost periodic) functions with compact range, and each member of both $\Lambda_\varkappa(\alpha(\gamma))$ and $\Lambda_\varkappa(\beta(\gamma))$ even has compact support. Moreover, setting

$$A(\gamma)(t) = \int_0^t \alpha(\gamma)(u)\,du \quad \text{for } t \in \boldsymbol{R}^+ \qquad \text{and} \qquad B(\gamma)(t) = \int_0^t \beta(\gamma)(u)\,du \quad \text{for } t \in \boldsymbol{R},$$

the indefinite integrals $A(\gamma)$ and $B(\gamma)$ also have compact range, but neither $A(\gamma)$ nor $B(\gamma)$ is weakly almost periodic in the sense of Eberlein.

Proof. To see that $A(\gamma)$ is not E.-w.a.p. on $\boldsymbol{R}^+$, let $\omega_n = 2^{2n+1}$, $n \in \boldsymbol{N}$, and put $t_m = 2^{2m}$, $m \in \boldsymbol{N}$. For $m, n \in \boldsymbol{N}$, if $m \leqslant n$, then $2^{2n+1}+1 \leqslant 2^{2n+1}+2^{2m} \leqslant 2^{2n+2}-$ so that $A(\gamma)(2^{2n+1}+2^{2m}) = \int_{-1}^{1} \gamma(u)\,du$. On the other hand, if $m \geqslant n+1$, then $2^{2m}+1 \leqslant 2^{2n+1}+2^{2m} \leqslant 2^{2m+1}-1$ whence $A(\gamma)(2^{2n+1}+2^{2m}) = 0$. Thus,

$$\lim_m \lim_n A(\gamma)_{\omega_n}(t_m) = \int_{-1}^{1} \gamma(u)\,du \qquad \text{and} \qquad \lim_n \lim_m A(\gamma)_{\omega_n}(t_m) = 0,$$

which suffices to establish our claim concerning $A(\gamma)$ and also shows that $B(\gamma)$ is not E.-w.a.p. on $\boldsymbol{R}$. The remaining assertions are either obvious or were considered in connection with Example 3.8. ■

Final remarks. Theorem 4.3 appears to offer a satisfactory solution to problem (0.b), and the same can be said for Theorem 4.9 with regard to the corresponding problem for integrals of w.a.a.p. functions. Except in the special case covered by Theorem 4.11, however, we have not been successful in determining exactly when the integral of an E.-w.a.p. function will again be E.-w.a.p.

One missing factor, seemingly, is an appropriate description of the critical part in the decomposition of E.-w.a.p. functions. As we have seen in this case, if $T \in \{\boldsymbol{R}, J_a\}$, then the critical part is taken by an E.-w.a.p. function $\phi \in C_b(T, X)$ for which $0 \in \Lambda_w(\phi)$. The question then arises as to whether these functions might coincide with the class treated in Theorem 4.12, but this is apparently not even known in the scalar case. Nonetheless, Theorems 4.11 and 4.12 do represent a positive step toward a conclusive solution of the basic problem.

Added in Proof

1. In this paper, the concept of Eberlein weak almost periodicity turned up naturally in our main problem of determining the analog of the Bohl–Bohr integration theorem for asymptotically almost periodic functions. However, this concept has its importance as well in the theory of semigroups of operators and the associated Cauchy problem. For the case of linear operators, this has been pointed out in Remark 1 following Lemma 4.6. In the nonlinear case, we have shown that the known strong ergodic limit theorems for nonlinear contraction semigroups in uniformly convex Banach spaces are consequences of the fact that the respective motions and almost-orbits of the semigroups in question actually are Eberlein weakly almost periodic [W. M. Ruess and

W. H. Summers, *Weak almost periodicity and the strong ergodic limit theorem for contraction semigroups*, Israel J. Math., to appear; and *Presque-périodicité faible et théorème ergodique pour les semi-groups de contractions non linéaires*, C. R. Acad. Sci. Paris 305 (1987), 741–744].

2. The concept of vector valued Eberlein weakly almost periodic functions, the corresponding special case of Theorem 2.1 for $C1_b(T, X)$, and a version of Lemma 3.3 have also been given by P. Milnes [J. London Math. Soc. 22 (1980), 467–472].

3. Since the submission of the manuscript, some of the problems left open in the text have been solved:

a. Uniform continuity of E.-w.a.p. functions (Lemma 3.1 and Remark following Theorem 3.2):

Given any Banach space X, every X-valued Eberlein weakly almost periodic function defined on $\boldsymbol{R}$, respectively on J_a, $a \in \boldsymbol{R}$, is (norm-) uniformly continuous [W. M. Ruess and W. H. Summers, *Ergodic theorems for semigroups of operators*, to appear].

b. Structure of the weak ω-limit set of an E.-w.a.p. function (Final remarks, second paragraph):

Even in the scalar case, there exist Eberlein weakly almost periodic functions f defined on $\boldsymbol{R}^+$ such that $0 \in \Lambda_w(f)$ but $\Lambda_w(f) \not\subset C_0(\boldsymbol{R}^+)$ [W. M. Ruess and F. D. Sentilles, *Weak mixing versus strong mixing, and a special class of weakly almost periodic functions*, to appear].

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